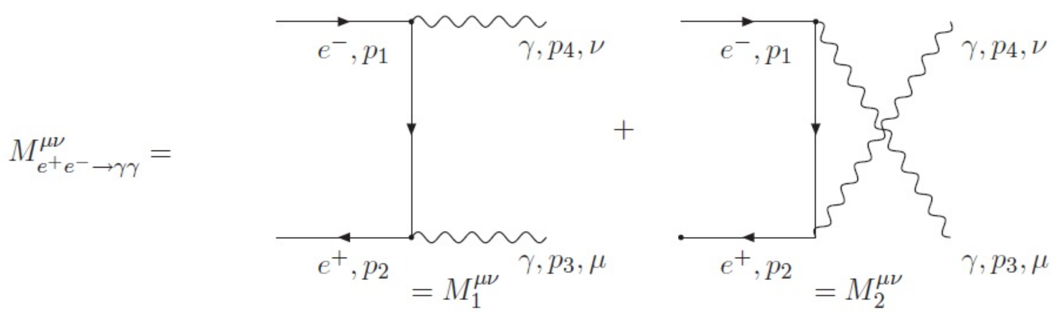
QCD

1. Consider electron-positron annihilation into photons in the context of pure QED. At tree-level there are only two Feynman diagrams which contribute to this process:



   - Compute $M^{\mu\nu}_{e^+e^-\to\gamma\gamma}$ (the amplitude without the polarization vectors).
   - Verify that the previously calculated amplitude fulfils current conservation (QED Ward's identity), i.e.

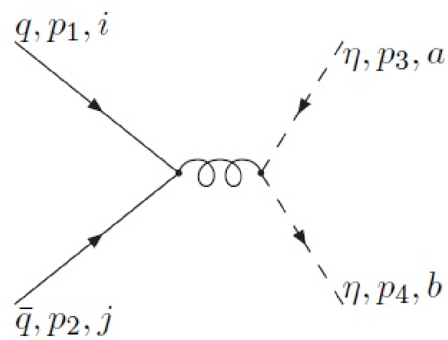
$$(p_3)_\mu \, M^{\mu\nu}_{e^+e^-\to\gamma\gamma} = 0 \quad (p_4)_\nu \, M^{\mu\nu}_{e^+e^-\to\gamma\gamma} = 0 \,. \tag{1}$$

2. Consider quark-antiquark annihilation into a gluon pair in the context of QCD. We know that QCD is a non-abelian gauge theory and that it is possible to get analytical expressions for three and four gluons vertices from the lagrangian. The aim of this exercise is to motivate a physical derivation of the triple-gluon vertex. Perform the calculation in a covariant gauge (in other words, use $d_{\mu\nu} = -g_{\mu\nu}$).

   - Taking into account only the quark-gluon interaction, draw all the possible tree-level Feynman diagrams associated with $q\bar{q} \to gg$ and compute $(M^1)^{\mu\nu}_{q\bar{q}\to gg}$ (use the results from the previous exercise)
   - Test the validity of current conservation in this case, contracting $(M^1)^{\mu\nu}_{q\bar{q}\to gg}$ with p_3 and p_4. Explain the result.

- The triple-gluon interaction should be proportional to f_{abc} ($SU(3)$ structure constant) times a pure kinematical factor $V^{\mu\nu\rho}(k_1, k_2, k_3)$, with k_i momenta of the interacting gluons. Write $V^{\mu\nu\rho}(k_1, k_2, k_3)$, assuming that it is the most general combination of four-vectors and metric-tensors which is globally antisymmetric and linear in the momenta.
- Assuming that the triple gluon vertex is $g_s V_0 f_{abc} V^{\mu\nu\rho}(k_1, k_2, k_3)$ (with V_0 a numerical constant), compute the associated diagrams and add this contribution to $(M^1)^{\mu\nu}_{q\bar{q}\to gg}$ to get $M^{\mu\nu}_{q\bar{q}\to gg}$. Again, test current conservation, without making any assumption about gluon polarization. What is wrong this time? Which conditions should be imposed in order to get Ward's identity fulfilled?
- Show that in $q\bar{q} \to gg$ the sum over non-physical polarizations is non-zero.
- Finally, compute the ghost contribution,

$M_{Ghost} =$



and verify that

$$M^{\mu\nu}_{q\bar{q}\to gg}(p_3)_\mu - (p_4)^\nu M^{\mathrm{Ghost}}{}_{q\bar{q}\to a\bar{a}} = 0 \tag{2}$$

Explain the result. Show that the ghost contribution exactly cancels the sum over non-physical polarizations.

3. Consider the process $gg \to gg$ at LO in QCD. Assuming that you only know the expression for quark-gluon and triple-gluon interactions, work out the four-gluon vertex following the ideas described in Exercises 1 and 2. Remember that this vertex has to be totally symmetric when exchanging particles and it has to be independent of particle momenta (why?).

4. Given a representation R of $SU(N)$, we know that its generators $T^a(R)$ obey the commutation rules

$$\left[T^a(R), T^b(R)\right] \;=\; \imath f^{abc} T^c(R)\,, \tag{3}$$

with f^{abc} the structure constant of $SU(N)$. Also, for each representation we can build the quadratic Casimir operator $T^2(R) = T^a(R)T^a(R)$, which verifies $T^2(R) = C_R \, \mathrm{Id}_R$ since it commutes with every generator.

- Using Jacobi identity $[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0$, show that the generators of the adjoint representation, $(T^a)_{bc} = \imath f_{abc}$ satisfy the commutation rule.
- Demonstrate the following Fierz identity

$$t^a_{ij} t^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}\right) \tag{4}$$

  with t^a_{ij} generators of $SU(N)$ algebra.
- Generators of a representation are usually normalised according to $\mathrm{Tr}\left[T_a(R)T_b(R)\right] = T_R\delta_{ab}$. Using $T_F = 1/2$ and $T_A = N$ show that

$$C_F = \frac{N^2-1}{2N} \quad , C_A = T_A = N\,. \tag{5}$$

  (Use for this item the Fierz identity.)

5. Starting from the fact that any QCD amplitude can be written as the product of a purely kinematical part times a colour coefficient, compute the colour factors that appear in the following squared matrix elements at LO:

   - $\left|\mathcal{M}\left(qq' \to qq'\right)\right|^2$
   - $\left|\mathcal{M}\left(q\bar{q} \to q\bar{q}\right)\right|^2$
   - $\left|\mathcal{M}\left(qg \to qg\right)\right|^2$
   - $\left|\mathcal{M}\left(q\bar{q} \to gg\right)\right|^2$
   - $\left|\mathcal{M}\left(gg \to gg\right)\right|^2$

6. **Renormalization schemes:** Let's start with the definition of Λ_{QCD}

$$\log\left(\frac{Q^2}{\Lambda^2_{QCD}}\right) = -\int_{\alpha_S(Q)}^{\infty} \frac{dx}{\beta(x)} \tag{6}$$

   where the β-function is given by

$$\beta\left(\alpha_S\right) = \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} = -b\alpha_S^2\left[1 + b'\,\alpha_S + b''\,\alpha_S^2 + \mathcal{O}\left(\alpha_S^3\right)\right]\,. \tag{7}$$

- Consider two renormalization schemes A and B, where the couplings are related by

$$\alpha_S^B = \alpha_S^A \left[1 + c_1\alpha_S^A + c_2(\alpha_S^A)^2 + \mathcal{O}\left((\alpha_S^A)^3\right)\right] . \tag{8}$$

  Show that the first two coefficients b and b' are scheme-independent, whereas the third one satisfies

$$b''_B - b''_A = c_2 - b'c_1 - c_1^2 . \tag{9}$$

- Show that the scale parameters of the two schemes are related by

$$\frac{(\Lambda_{QCD})_A}{(\Lambda_{QCD})_B} = \exp\left(\frac{c_1}{2b}\right) . \tag{10}$$

  (*Hint*: Use the definition of Λ_{QCD} given in the first item and then take the limit $Q \to \infty$, $\alpha_{A,B} \to 0$.)

7. $e^+e^- \to q\bar{q}g$

- Show that the phase space for the unpolarized decay into three massless objects can be written as:

$$dPS_3 = \frac{1}{(2\pi)^5}\frac{s}{32}dx_1 dx_2 d\alpha d(\cos\beta)d\gamma \tag{11}$$

  where s is the c.m.s. energy, $x_i = 2E_i/\sqrt{s}$ are the fractional energies of the partons and α, β and γ represent the angles needed to describe the system.

- Calculate the matrix element squared for $e^+e^- \to q\bar{q}g$. After angular integrations show that one can write

$$|M|^2 = \frac{1}{s^2}L^{\mu\nu}H_{\mu\nu} \to \frac{1}{s^2}(L^{\mu\nu}g_{\mu\nu})(H^{\rho\sigma}g_{\rho\sigma}) \tag{12}$$

  where $L^{\mu\nu}$ and $H^{\mu\nu}$ are the leptonic and hadronic tensors arising from the corresponding currents. The result is

$$\sigma^{q\bar{q}g} = \sigma^{LO} C_F \frac{\alpha_s}{2\pi}\int dx_1 dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \tag{13}$$

  where $\sigma^{LO} = N_c(\sum_f Q_f^2)4\pi\alpha^2/(3s)$

- Identify the origin of the singular structure of the previous expression.

- Perform the same calculation for a scalar gluon and verify that (apart from an overall normalization)

$$\sigma^{q\bar{q}s} = \sigma^{LO}\int dx_1 dx_2 \frac{x_3^2}{2(1-x_1)(1-x_2)} \tag{14}$$

8. Compute the real contribution to the splitting functions (and fix the virtual by using sum rules). There are many ways to do it, you can take the matrix element for a physical process and evaluate the corresponding collinear limit or perform a direct calculation. You can obtain a couple of kernels (those initiated by quarks) by analysing the collinear limits of the previous exercise! In order to compute those initiated by gluons you can look at processes involving the Higgs boson. You can use the following matrix elements, obtained using the effective coupling between the gluons and the Higgs boson:

$$|\mathcal{M}_{H\to g\ g\ g}|^2 = \frac{\alpha_s^3}{v^2}\frac{32}{3\pi}\frac{M_H^8 + s_1^4 + s_2^4 + s_3^4}{s_1 s_2 s_3} \tag{15}$$

and

$$|\mathcal{M}_{H\to g\ q\ \bar{q}}|^2 = \frac{\alpha_s^3}{v^2}\frac{16}{9\pi}\frac{s_2^2 + s_3^2}{s_1}\,, \tag{16}$$

where ($\overline{\text{be careful!}}$)

$$\overline{|\mathcal{M}_{gg\to H}|}^2 = \frac{\alpha_s^2}{\pi^2}\frac{M_H^4}{576 v^2}\,, \tag{17}$$

and $s_i = (P_H - P_i)^2$ with P_H and P_i the Higgs and corresponding parton four-momenta, respectively.

Afterwards, read the original paper by Altarelli and Parisi (Asymptotic Freedom in Parton Language, Altarelli and Parisi, Nuclear Physics B126 (1977) p.298) and promise yourself you will make an attempt to write a paper like that in your lifetime as a physicist! (and don't be upset if you don't manage to do it...)

9. The rapidity y and pseudo-rapidity η are defined as

$$\begin{aligned} y &= \frac{1}{2}\log\left(\frac{E+p_z}{E-p_z}\right) \\ \eta &= -\log\left(\tan\left(\frac{\theta}{2}\right)\right) \end{aligned} \tag{18}$$

where the z direction is that of the collider beams.

- Verify that for a particle of mass m

$$\begin{aligned} E &= \sqrt{m^2 + p_T^2}\cosh y \\ p_z &= \sqrt{m^2 + p_T^2}\sinh y \\ p_T^2 &= p_x^2 + p_y^2 \end{aligned} \tag{19}$$

- Prove that $\tanh \eta = \cos\theta$
- Prove that $\eta = y$ for a relativistic particle ($E \gg m$).
- Prove that the phase space factor for one particle is

$$\frac{d^3p}{E} = p_T \, dp_T dy d\phi \tag{20}$$

10. Consider a generic particle X of mass M (such as a Z boson or a Higgs) produced on shell at the LHC , with zero transverse momentum, $pp \to X$. Find the relevant values of x_1, x_2 of the initial partons that can be accessed by producing such a particle.

11. At the LHC, partons in the incoming beams (beam energy $E = 7$ TeV) collide with a momentum fraction x_1, x_2 and produce two jets with negligible mass, transverse momentum p_T and rapidities $y_{3,4}$. Show that

$$x_1 = \frac{p_T}{\sqrt{s}}(e^{y_3} + e^{y_4}), \qquad x_2 = \frac{p_T}{\sqrt{s}}(e^{-y_3} + e^{-y_4}) \tag{21}$$

Also show that the invariant mass of the dijet system is

$$M_{JJ} = 2p_T \cosh\left(\frac{y_3 - y_4}{2}\right) \tag{22}$$

and the centre-of-mass scattering angle is given by

$$\cos\theta^* = \tanh\left(\frac{y_3 - y_4}{2}\right) \tag{23}$$

what is the meaning of $\frac{y_3+y_4}{2}$?

12. Compute the cross section for the production of the Higgs boson at hadronic colliders in the dominant channel $gg \to H$ at the lowest order

13. Show that in DIS mediated by photon exchange, the hadronic tensor can be decomposed as

$$F_1\left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right) + F_2\frac{1}{p\cdot q}\left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right)$$

How does that change in case of weak interactions?

14. Analyse the soft limit of QED and QCD amplitudes (photon and gluon emission, respectively) and obtain the corresponding expression for the eikonal factor.